Periodic cycle length in the sine circle map

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Abstract

Nonlinear finite difference equations exhibit fascinating periodic behaviour. We examine dynamics of the sine circle map $x_{t+1} = x_t + a + b \sin(2\pi x_t) \pmod 1$. The occurrence of periodic cycles in this system and their length vary with the function's parameters a, b, and with x_t. We propose a concise method to graphically represent these dynamics in *Mathematica*.

Introduction

Complex phenomena occurring over time can be described with the mathematical tools of nonlinear dynamics. Finite-difference equations relate values at discrete times. They take the general form $N_{t+1} = f(N_t)$, where N_t denotes the state of the system at time t. The change of this state in time are the dynamics of the system (Kaplan & Glass, 1995, p. 2). In nonlinear finite difference equations such as the sine circle map, the dynamics of the system are dependent on the interaction of two or more variables (p. 8). In such systems, a range of different behaviours are possible, including steady states where a solution approaches a certain state and remains fixed there; chaos, where the solution oscillates in a non-periodic manner; and periodic cycles (p. 9-11).

Periodic cycles are patterns that occur repeatedly after a given number of iterations, i.e. the number of iterations between a repetition is the period of the cycle (p. 20-21). For a finite-difference equation of the form $N_{t+1} = f(N_t)$, cycles arise when

$$x_{t+n} = x_t, \qquad \text{but} \qquad x_{t+j} \neq x_t \qquad \text{for} \qquad j = 1, 2, \ldots, n-1.$$

Periodic cycles can generally be found using a link between fixed points and cycles: if a system $x_{t+1} = f(x_t)$ has a cycle of period two, then the function $f(f(x_t))$ has at least two fixed points; consequently cycles of period two can be found by solving the equation $x_t = f(f(x_t))$. In a generalized form, this means that a cycle of period n is given by the n^{th} solution to the equation $x_t = f(f(\ldots f(x_t)))$.

Cycles can be locally stable, i.e. starting from an initial condition close to one of its solutions, subsequent iterates will approach the cycle (p. 23). Such local asymptotic stability can be found by investigating the slope of the respective function at its solutions, e.g. for cycles of period two:

$$\frac{df\left(f\left(x_t\right)\right)}{dx_t}|x^{¿}=\frac{df}{dx_t}|f\left(x^{¿}\right)\ \frac{df}{dx_t}|x^{¿}$$

So for there to be a stable cycle of period n, there must be a minimum number of n fixed points at each of which the slope of the function is equal and has an absolute value of less than 1.

As a function's parameters change, a finite-difference equation can change from one form of qualitative behaviour to another, undergoing a so-called bifurcation (p. 29). In particular, many nonlinear systems exhibit period-doubling bifurcation, a sequence of bifurcations in which the period of its cycles doubles abruptly as a parameter changes. Such a period-doubling bifurcation typically occurs when the slope of the function is -1 for some parameter. In this case, the periodic cycle of period n loses stability, and a periodic cycle of period $2n$ gains stability, i.e. the period of the stable cycles doubles (ibid.). This change in the length of cycles' periods with changing parameters can be displayed in a bifurcation diagram.

In this paper, we analyse a nonlinear finite-difference equation with two parameters a and b known as the sine circle map,

$$x_{t+1} = x_t + a + b\sin(2\pi x_t) \quad (\mathrm{mod}\ 1) \quad \text{where} \quad 0 \le a \le 1,\ 0 \le x_t \le 1, \text{ and } b \ge 0.$$

We are studying how the periodic orbits of the sine circle map change as a function of a and b. In the following, we will elaborate on the graphical representation of these dynamics.

Methods

Mathematica (Wolfram Research, 2010) was used to determine the lengths of periodic cycles in the sine circle map for varying values of the parameters a and b (Fig. 1). This was done for random initial values $0 \le x_t \le 1$.[1] As a restriction, the values for b were limited to $b \le 10$ due to constraints in computing power. The program was set to iterate the function 10 times. If a repetition of the initial value is found during these iterations, the program lists the respective coordinates. This creates a list of coordinates on the x,y-plane with their associated periodic cycle lengths.

The resulting list of coordinates is displayed on a 3D plot, where the a and b parameters serve as the x- and y-axis, respectively. The length of the periodic cycle at each coordinate is

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[1] A slider for x_t could not be included due to constraints in computing power.

displayed on the z-axis, and the differences in cycle length are also emphasized by a colour gradient from blue (short) to yellow (long).

A density plot is added that presents a top view of the 3D plot, with the colour gradients indicating the length of periodic cycles. Again, parameter a lies on the x-axis and parameter b on the y-axis.

```
initialvalue = x_t;
CycleN :=
 Position[NestList[Mod[Round[# + a + b*Sin[2*Pi*#], 0.01], 1] &,
   initialvalue + a + b*Sin[2*Pi*initialvalue], 10], initialvalue, 1,
  1]
ListPlot3D[
 Table[If[CycleN != {}, Extract[Extract[CycleN, {1}], {1}], 0], {a, 0,
     1, 0.01}, {b, 0, 10, 0.01}], AxesLabel -> {a, b},
 ColorFunction -> "BlueGreenYellow"]
ListDensityPlot[
 Table[If[CycleN != {}, Extract[Extract[CycleN, {1}], {1}], 0], {a, 0,
     1, 0.01}, {b, 0, 10, 0.01}], AxesLabel -> {a, b},
 ColorFunction -> "BlueGreenYellow"]
```

Fig. 1: Plotting the lengths of the periodic cycles of the sine circle map as a function of its parameters a, b.

Second, we also plotted the finite-difference equation for the sine circle map as a function of x_{t+1} and x_t. This gives an indication of the effects of the function's parameters on its fixed points.

```
f[x_, a_, b_] := Mod[{x + a + b*Sin[2*Pi*x]}, 1]
Manipulate[
 Plot[{f[x, a, b], x}, {x, 0, 1},
  AxesLabel -> {x, Subscript[x, t + 1]}], {a, 0, 1}, {b, 0, 10}]
```

Fig. 2: Plotting the sine circle map as a function of x_{t+1} *and* x_t

Results

We analyse a finite differential equation with two parameters, also known as the sine circle map:

$$x_{t+1} = x_t + a + b\,sin(2\pi x_t) \quad (mod\ 1) \quad \text{where} \quad 0 \leq a \leq 1,\ 0 \leq x_t \leq 1, \text{ and } 0 \leq b \leq 10.$$

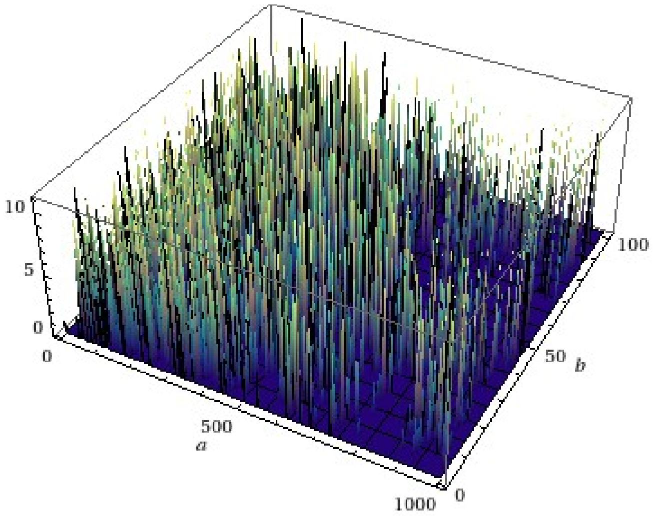
The mod constraint ensures that if the value of x_t exceeds that of the modulus - 1 in this case -, x_t will be continually subtracted by 1 until $0 \leq x_t \leq 1$.

When this function is plotted with the parameters a and b left as variable (Fig. 2), it becomes clear that the function has three stable fixed points: one at the origin (0,0), another at

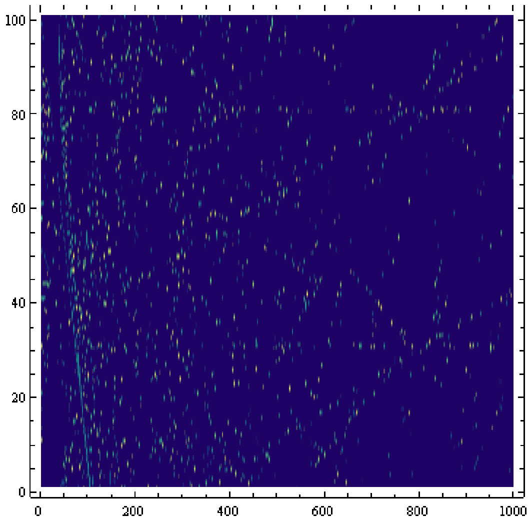
(0.5,0.5) and the last at (1,1). Additionally, as *b* is increased, the amplitude of the sine circle map increases. However, this is counteracted by the modulus, which effectively cuts the sine curve and moves the sections back into the range of $0 \leq x_{t+1} \leq 1$. Thus, as *b* increases, further unstable fixed points are added. If *a* increases, the amount of fixed points remains the same, but their coordinates vary. This is because *a* only causes vertical translation, which becomes horizontal translation due to the modulus.

The behaviour of the sine circle map as a function of its parameters *a*, *b* is more complex. Due to constraints in computing power, we could only create a limited number of plots for a range of values for x_t, which exhibit diverse behaviour. Nevertheless, these plots not only allow us to find particular behaviours of a specific parameter, but also enable us to understand patterns in relation to the parameters. We will discuss some examples in the following.

For an initial condition of $x_t = 0.3$, (Ill. 1) and (Ill. 2) give the 3D plot and the density plot of periodic cycle length as a function of the parameters *a* and *b*. These plots show that at least for this initial condition, periodic cycles of a length ≤ 10 (the maximum length displayed) become rarer as *a* increases, whereas the variation in *b* has little impact on the density of periodic cycles. Also, for large *a* (≥ 0.6), spiderweb-shaped patterns emerge. This might indicate a period-doubling “road to chaos” (Kaplan & Glass, 1995, p. 29).

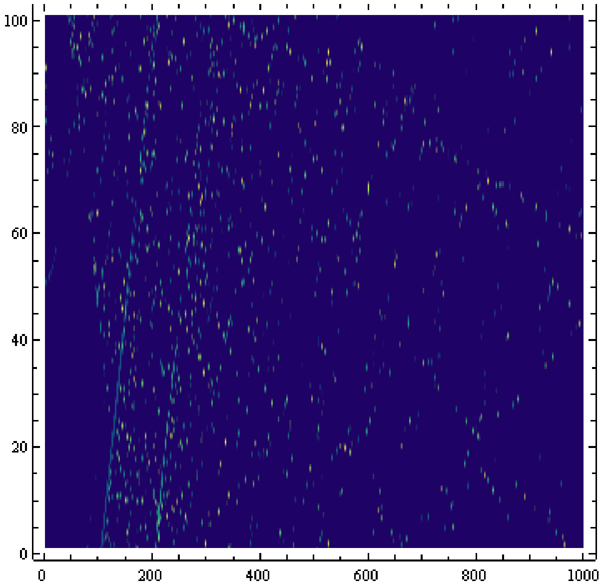


Ill. 2: 3D Plot for initial condition $x_t = 0.3$

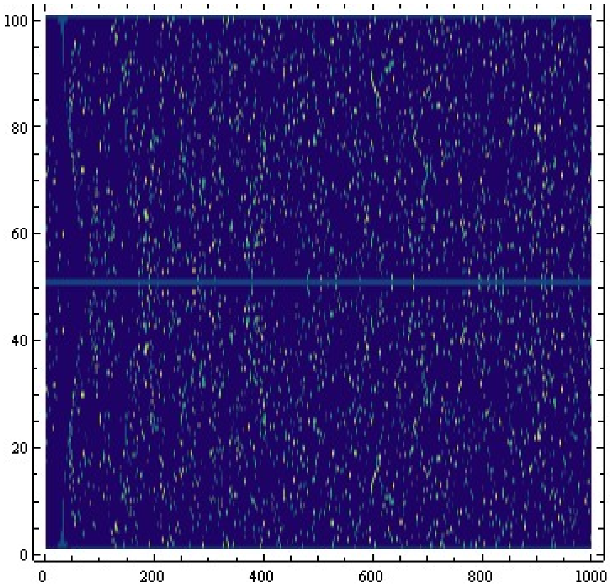


Ill. 1: Density Plot for initial value $x_t = 0.3$

Another example, (Ill. 3) shows that for an initial condition $x_t = 0.7$ and a parameter value $0 \leq a \leq 0.1$ there will not be many cycles, implying that for these values the function goes to either a fixed point or chaos regardless of the *b*-value. In the range of $0.1 \leq a \leq 0.3$, we will have a higher quantity of cycles for every value of b that will become scarcer as b increases. After $a \geq 0.3$, the periodicity of the function will decrease for every value of b.



Ill. 3: Density Plot for initial value $x_t = 0.7$



Ill. 4: Density Plot for initial value $x_t = 0.5$

The differences between plots illustrate the importance of the initial condition for the subsequent dynamics. This becomes even more clear in (Ill. 4). In this plot for $x_t = 0.5$ there is a fairly uniform periodic behaviour A striking finding is the thick line that crosses the plot by the middle. This line means that for a value $b \approx 5$, there is a cycle of the same period length for all values of a Around this line, the plot is symmetric, i.e. the same pattern of periodicity is exhibit for values of b above and below 5.

Conclusion

We have investigated the dynamics of the nonlinear finite-difference equation known as sine circle map. The behaviour of this equation is dependent on the function's two parameters a, b as well as the initial condition of the system x_t. The occurrence of periodic cycles in this system and their length depends on both parameters and the variable. We present a concise method to graphically represent the dynamics of the system, and in particular periodic cycle length as a function of the parameters a, b using *Mathematica*.

An examination of plots for varying initial conditions x_t shows that the length of periodic cycles varies with both a, b, but also in dependence on the initial condition. A variety of patterns emerge, including symmetries.

References

Kaplan D, Glass L. 1995. *Understanding Nonlinear Dynamics*. New York, NY: Springer-Verlag.

Wolfram Research. 2010. *Mathematica* (v8.0). Champaign, IL: Wolfram Research.